The random bond Ising model on the Bethe lattice

1986 J. Phys. A: Math. Gen. 19 3883

(http://iopscience.iop.org/0305-4470/19/18/033)

J. Phys. A: Math. Gen. 19 (1986) 3883-3890. Printed in Great Britain

The random bond Ising model on the Bethe lattice

D S McKenzie and M A S Saqi†

Department of Physics, Kings College, University of London, Strand, London WC2R 2LS, UK

Received 22 January 1986

Abstract. This paper studies the random bond Ising model of a spin glass on a tree of coordination number $q=3$. The model is studied as an example of the method of ring recurrence. A set of recursion relations between the moments of the probability distribution of the order parameter are produced. It is shown that solutions of the recursion relations exist which form an infinite bifurcating set below a critical temperature τ_c. The nature of the first such solution is investigated in detail and the critical exponent analogous to β is found.

1. Introduction

In this paper we study the random bond Ising model of a spin glass as an example of a discrete physical system (G, Φ, U) (McKenzie 1981) using the method of ring recurrence (McKenzie 1986). The model is defined by (i) the graph G for which we have chosen the $q=3$ tree, q being the coordination number or local degree of each vertex (the model is easily generalised to arbitrary q), (ii) the state space $\Phi=\Lambda\times\Psi$ where $\Lambda=\varphi^{|V(G)|}$, $\Psi=\psi^{|E(G)|}$ with $V(G)$ and $E(G)$ the vertex and edge set of G respectively, $\varphi=\{1,-1\}$ and ψ is the set of real numbers $\mathbb{R}$ (or a subset of $\mathbb{R}$), and (iii) the potential given by

$$U=\beta H\sum_{x\in V(G)}\sigma_x+\beta\sum_{[x,y]\in E(G)}J_{[x,y]}\sigma_x\sigma_y$$

where $\beta=(kT)^{-1}$ is included in the potential, $H\in\mathbb{R}$, $J_{[x,y]}\in\psi$ and σ_x, $\sigma_y\in\varphi$. Each $J_{[x,y]}$ occurs with probability $P(J_{[x,y]})$, such that there is a mapping

$$P:\psi\to[0,1]\qquad\sum_{z\in\psi}P(z)=1.$$

Each state $\omega\in\Phi$ is assigned a probability

$$p(\omega)=p(\omega_1/\omega_2)p(\omega_2)$$

where $\omega\in\Lambda$, $\omega_2\in\Psi$ and $p(\omega_1/\omega_2)$ is the conditional probability that the spin state ω_1 occurs, given the distribution of edge interaction energies ω_2. It is assumed that the edge interaction energies are assigned independently so that

$$p(\omega_2)=\prod_{[x,y]\in E(G)}P(J_{[x,y]}).$$

† Present address: Department of Mathematics, Imperial College of Science and Technology, London SW7 2BZ, UK.

0305-4470/86/183883+08$02.50

We also assume that the conditional probabilities $p(\omega_1/\omega_2)$ are determined by a canonical distribution

$$p(\omega_1/\omega_2) = \exp[U(\omega_1, \omega_2)]/Z(\omega_2)$$

with

$$Z(\omega_2) = \sum_{\omega_1} \exp[U(\omega_1, \omega_2)].$$

This model is the so-called quenched model. It corresponds physically to a collection of microcrystallites, each of whch is small on a macroscopic scale but large on the scale of atomic bond lengths. Each microcrystallite has a particular assignment of edge interaction energies so that the magnetisation of each microcrystallite m is given in the usual way by

$$m(\omega_2) = \partial \log Z(\omega_2)/\partial H$$

but the magnetisation of the system as a whole, M, is an average over all the assignments of edge interaction energies, i.e.

$$\langle M \rangle = \sum_{\omega_2} m(\omega_2) p(\omega_2).$$

The potential of our model is the Sherrington-Kirkpatrick Hamiltonian (1975), dressed up slightly differently. A partial solution to our model with $p(z)$ chosen as Gaussian has been described by Thompson (1982). In the study we produce an exact solution for the magnetisation which shows the existence of an infinite number of stable non-trivial phases below a critical temperature. Our method does not use the replica trick, and as mentioned earlier the solution is easily generalised to arbitrary values of q. Furthermore, any properly normalised symmetric function can be chosen for $p(z)$ without affecting the solution, other than to change the values of the critical points. For concreteness we later choose $p(z)$ to be rectangular, i.e.

$$p(z) = 1/2a \qquad z \in [-a, a]$$

but this assumption is not essential to our argument.

2. Derivation of the recursion relations

The general method of ring recurrence is described in McKenzie (1986). We consider here the calculation of the conditional partition function $Z(\omega_2)$ for a fixed distribution of edge interaction energies. We divide the graph G into rings X_s^α relative to some arbitrary origin, $\alpha \in V(G)$. Here $X_s^\alpha = \{x: x \in V(G), \mathrm{d}(x, \alpha) = s\}$ where $\mathrm{d}(x, \alpha)$ is the usual graph metric and s is a positive integer. Let us define recursively the set of edge interaction energies $\mathscr{X}(x_i^{(s)}, x_j^{(s+1)})$, where $x_i^{(s)} \in X_s^\alpha$, $x_j^{(s+1)} \in X_{s+1}^\alpha$ and $[x_i^{(s)}, x_j^{(s+1)}] \in E(G)$, by

$$\mathscr{X}(x_i^{(s)}, x_j^{(s+1)}) = \{J_{[x_k, y_k]}: [x_k, y_k] \in E(G), x_k \in X_{s+k}^\alpha, y_k \in X_{s+k+1}^\alpha, \\ k = 0, 1, 2, \ldots, x_0 = x_i^{(s)}, y_0 = x_j^{(s+1)}\}.$$

We now define effective fields $\tau_1^{(s-1)}$ and $\tau_2^{(s-1)}$ by

$$\lambda_{s-1}\tau_1^{(s-1)}(\mathscr{X}(x_i^{(s-1)}, x_j^{(s)})) \\ = \exp(\beta H + \beta J_{[x_i^{(s-1)}, x_j^{(s)}]})\tau_1^{(s)}(\mathscr{X}(x_j^{(s)}, x_1^{(s+1)}))\tau_1^{(s)}(\mathscr{X}(x_j^{(s)}, x_2^{(s+1)}))$$

$$+\exp(-\beta H-\beta J_{[x_i^{(s-1)},x_j^{(s)}]})\tau_2^{(s)}(\mathscr{R}(x_j^{(s)},x_1^{(s+1)}))\tau_2^{(s)}(\mathscr{R}(x_j^{(s)},x_2^{(s+1)}))$$

$$\lambda_{s-1}\tau_2^{(s-1)}(\mathscr{R}(x_i^{(s-1)},x_j^{(s)}))$$

$$=\exp(\beta H-\beta J_{[x_i^{(s-1)},x_j^{(s)}]})\tau_1^{(s)}(\mathscr{R}(x_j^{(s)},x_1^{(s+1)}))\tau_1^{(s)}(\mathscr{R}(x_j^{(s)},x_2^{(s+1)}))$$

$$+\exp(-\beta H+\beta J_{[x_i^{(s-1)},x_j^{(s)}]})\tau_2^{(s)}(\mathscr{R}(x_j^{(s)},x_1^{(s+1)}))\tau_2^{(s)}(\mathscr{R}(x_j^{(s)},x_2^{(s+1)})) \tag{2.1}$$

(see figure 1). The functional form of the above equations is independent of the choice of vertices $x_i^{(s-1)}, x_j^{(s)}$. To simplify the notation let us write $[x_i^{(s-1)}, x_j^{(s)}]$ as $e^{(s-1)}$, $[x_j^s, x_1^{(s+1)}]$ as $e_1^{(s)}$ and $[x_j^{(s)}, x_2^{(s+1)}]$ as $e_2^{(s)}$. We choose the norm λ_{s-1} by requiring $\tau_1^{(s-1)}\equiv 1$ for all s and define $\mu^{(s)}=\tau_2^{(s)}/\tau_1^{(s)}$ to obtain a single recursion relation

$$\mu^{(s-1)}(\mathscr{R}(e^{(s-1)}))$$

$$=\frac{\exp(\beta H-\beta J_{e^{(s-1)}})+\exp(-\beta H+\beta J_{e^{(s-1)}})\mu^{(s)}(\mathscr{R}(e_1^{(s)}))\mu^{(s)}(\mathscr{R}(e_2^{(s)}))}{\exp(\beta H+\beta J_{e^{(s-1)}})+\exp(-\beta H-\beta J_{e^{(s-1)}})\mu^{(s)}(\mathscr{R}(e_1^{(s)}))\mu^{(s)}(\mathscr{R}(e_2^{(s)}))}. \tag{2.2}$$

The partition function is then given by

$$Z(\omega_2)=\left(\prod_{s\geqslant 1}\prod_{e^{(s)}\in E_{s,s+1}}\lambda_s(e^{(s)})\right)$$

$$\times\left(\prod(e^{\beta H}+e^{-\beta H}\mu^{(1)}(\mathscr{R}(e_1^{(1)}))\mu^{(1)}(\mathscr{R}(e_2^{(1)}))\right) \tag{2.3}$$

where $E_{s,s+1}=\{[x,y]: x\in X_s^\alpha, y\in X_{s+1}^\alpha\}$ and the norms are given by

$$\lambda_s(e^{(s)})=\exp(\beta H+\beta J_{e^{(s)}})+\exp(-\beta H-\beta J_{e^{(s)}})\mu^{(s+1)}(\mathscr{R}(e_1^{(s+1)}))\mu^{(s+1)}(\mathscr{R}(e_2^{(s+1)})).$$

It is convenient to introduce new fields $p^{(s)}$ given by $\mu^{(s)}=\exp(-2p^{(s)})$. The magnetisation per site of the system can be defined by the magnetisation of the central site which is given by

$$m(\omega_2)=\partial\log Z(\omega_2)/\partial H$$

where H is the field occurring explicitly in (2.3). In zero field we obtain

$$m(\omega_2)=\tanh(p^{(1)}(\mathscr{R}(e_1^{(1)}))+p^{(1)}(\mathscr{R}(e_2^{(1)}))).$$

Clearly for any given assignment of the bond strengths the magnetisations might have any value. However, the mean magnetisation $\langle M\rangle$ is well defined. This mean value is determined by the mean value of the $\tanh p^{(1)}$, which are determined recursively in terms of the $p^{(s)}$ by

$$\tanh p^{(s-1)}(\mathscr{R}(e^{(s-1)}))=\tanh\beta J_{e^{(s-1)}}\tanh(\beta H+p^{(s)}(\mathscr{R}(e_1^{(s)}))+p^{(s)}(\mathscr{R}(e_2^{(s)}))) \tag{2.4}$$

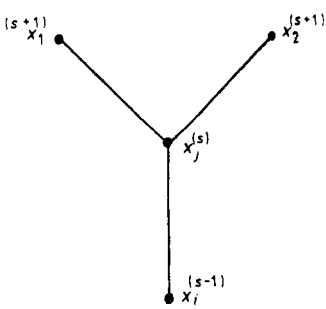

Figure 1. Refer to the text.

after some algebra. We transform (2.4) into a recursion relation between the average quantities simply by averaging the distribution of bond strengths. We use the notation

$$\langle \cdot \rangle = \sum_{\omega_2} (\cdot) p(\omega_2).$$

Since we shall require powers of $\tanh p$, we take the nth power of (2.4) and average to obtain

$$\langle \tanh^n p^{(s-1)} \rangle = \langle \tanh^n \beta J \tanh^n (\beta H + p^{(s)}(\mathscr{X}(e_1^{(s)})) + p^{(s)}(\mathscr{X}(e_2^{(s)}))) \rangle \tag{2.5}$$

for $n = 0, 1, 2, \ldots$. Let us now write

$$M_n^{(s)} = \langle \tanh^n p^{(s)}(\mathscr{X}(e^{(s)})) \rangle$$

and make the important observation that the dependence of $M_n^{(s)}$ on $\mathscr{X}(e^{(s)})$ is removed by averaging over all the bond strengths. Equation (2.5) is an implicit relation between $M_n^{(s-1)}$ and $M_m^{(s)}$, $m = 0, 1, 2, \ldots$. Because the same functional form of the recursion relation holds for all s, we can drop the superscripts on the moments $M_n^{(s)}$, i.e. we shall write $M_n^{(s)} = M_n$ for all $s \geqslant 1$. This is equivalent to considering the bulk properties only, neglecting edge effects. We therefore obtain a set of recursion relations written formally as

$$M_n = f_n(\{M_0, M_1 \ldots\}) \qquad n = 0, 1, 2 \ldots$$

whose solution we consider in the next section.

3. Solution of recursion relations in zero field

If we are to obtain a symmetry breaking phase transition we require $H = 0$. In this, the interesting case, the recursion relations become

$$M_n = \left\langle \tanh^n \beta J_e \left(\frac{\tanh p(\mathscr{X}(e_1)) + \tanh p(\mathscr{X}(e_2))}{1 + \tanh p(\mathscr{X}(e_1)) \tanh p(\mathscr{X}(e_2))} \right)^n \right\rangle. \tag{3.1}$$

The only contribution to the average of the bond e is the term in $\tanh \beta J_e$ in the above. Hence averaging over this bond can be separated out. Let us therefore write

$$I_n = \sum_J \tanh^n \beta J P(J) \qquad J \in \psi \tag{3.2}$$

and expand the second term in (3.1) to obtain

$$M_n = I_n \sum_{r=0}^{\infty} \sum_{t=r}^{r+n} \frac{(-1)^r n(n+r-1)!}{(n-t+r)!(t-r)!r!} M_t M_{n+2r-t} \tag{3.3}$$

where we have used the property that

$$\langle \tanh^m p(\mathscr{X}(e_1)) \tanh^n p(\mathscr{X}(e_2)) \rangle = \langle \tanh^m p(\mathscr{X}(e_1)) \rangle \langle \tanh^n p(\mathscr{X}(e_2)) \rangle$$

since $\mathscr{X}(e_1)$ and $\mathscr{X}(e_2)$ are disjoint sets. We note that $M_0 = 1$, $|M_n| \leqslant 1$ and $I_n = 0$ for odd n if $p(z)$ is a symmetric function of z. The latter obervation requires $M_n = 0$ for odd n. Explicit expressions for M_2, M_4, M_6 and M_8 are given below:

$$\begin{aligned}
M_2 &= 2I_2(M_2 - 2M_2^2 + 3M_2M_4 - 4M_4^2 + 5M_4M_6 - 6M_6^2 + \ldots) \\
M_4 &= 2I_4(M_4 + 3M_2^2 - 16M_2M_4 + 10M_2M_6 + 30M_4^2 - 80M_4M_6 + \ldots) \\
M_6 &= 2I_6(M_6 + 15M_2M_4 - 36M_2M_6 - 60M_4^2 + 315M_4M_6 + 21M_2M_8 + \ldots) \\
M_8 &= 2I_8(M_8 + 28M_2M_6 + 35M_4^2 - 64M_2M_8 - 448M_4M_6 + \ldots).
\end{aligned} \tag{3.4}$$

Clearly, expressions (3.3) have a trivial solution $M_n = 0$, $n = 2, 4, 6, \ldots$. We study later perturbations about the trivial solution but first let us consider the M_n as being the moments of a probability distribution function $\mathcal{M}(x)$, $x \in \mathbb{R}$ namely the probability of finding a value of $\langle \tanh p \rangle$, which we are treating as the order parameter of the system, with a value x. The moment generating function of $\mathcal{M}(x)$ is

$$\phi(t) = \int_{-\infty}^{\infty} \mathcal{M}(x)\, e^{itx}\, dx$$

$$= \sum_{n \geq 0} M_n\, (it)^n / n! \tag{3.5}$$

whose inverse gives

$$\mathcal{M}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi(t)\, e^{-itx}\, dt. \tag{3.6}$$

In the trivial case the solution set $\Theta = \{M_0, M_2, M_4 \ldots\}$ becomes $\Theta_0 = \{1, 0, 0, \ldots\}$ so that $\phi(t) = 1$ and $\mathcal{M}(x) = \delta(x)$. Thus, as expected, the probability of finding a value of the order parameter different from zero is zero and the system behaves as a paramagnet. To study the effect of perturbations about the trivial solution we rewrite equations (3.3) formally as

$$g_n(\Theta) = K_n M_n - h_n(\Theta) = 0 \qquad n = 2, 4 \ldots \tag{3.7}$$

where

$$K_n = (1 - 2I_n)/2I_n \tag{3.8}$$

and the functions $h_n(\Theta)$ involve only quadratic terms such as $M_k M_l$.

These quadratic terms are a consequence of the local degree q of each vertex being three. Similarly the factor $2I_n$, which occurs in (3.8), can be generalised to $(q-1)I_n$. The stability of the trivial solution $\Theta_0 = \{1, 0, 0 \ldots\}$ is investigated by forming the derivatives

$$(\partial g_n / \partial M_k)_{\Theta = \Theta_0} = K_n \delta_{nk} \tag{3.9}$$

where δ_{nk} is the Kronecker delta. The matrix of these derivatives is the Hessian H of the system. Clearly, along the locus of the trivial solution, H is diagonal and the eigenvalues are K_n. The trivial solution bifurcates whenever $\det H = 0$, i.e. at each point $K_n = 0$. Through the I_n, the K_n are functions of temperature. For even values of n

$$I_n = \sum_J \tanh \beta J P(J) < I_{n-2} \qquad n \geq 2$$

which implies $K_{n-2} < K_n$. Thus the trivial solution is stable for $K_2 > 0$ and the first bifurcation occurs at $K_2 = 0$. The trivial solution is unstable for all $K_2 < 0$.

The K_n are implicitly functions of temperature and decrease with decreasing temperature. To see this explicitly, let us consider the rectangular distribution for $P(z)$, $P(z) = 1/2a$, $z \in [-a, a]$. Then

$$I_n = \frac{1}{2a} \int_{-a}^{a} \tanh^n \beta J\, dJ.$$

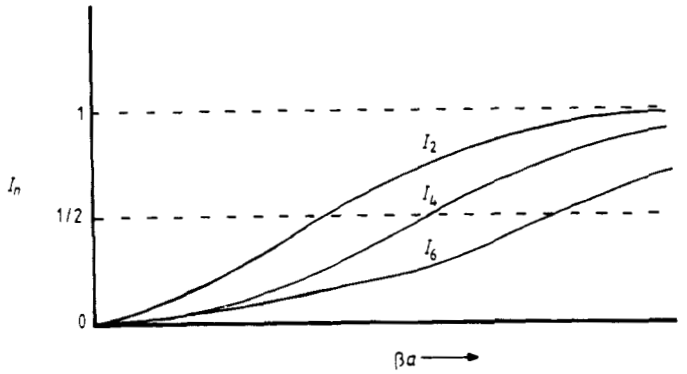

Figure 2. Schematic diagram of the variation of I_n with temperature, showing that each bifurcation occurs at increasingly lower temperatures as n increases.

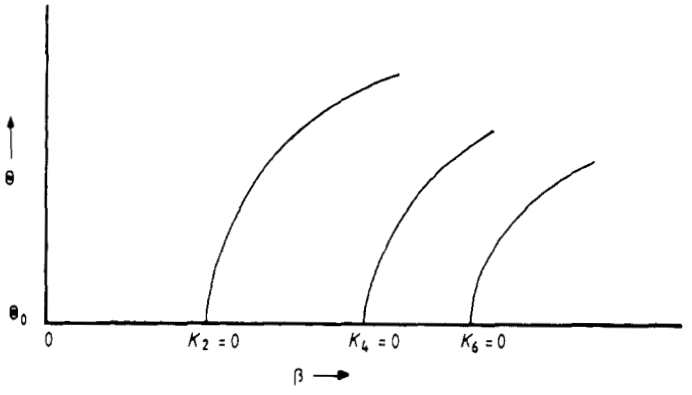

Figure 3. Schematic diagram of the set of bifurcating solutions as a function of temperature. This is a one-dimensional representation of the solution space of Θ. Each branch should start in a different plane orthogonal to all the others.

Using the recursion formula for $\int \tanh^n z \, dz$ (Abramowitz and Stegun 1965) and the series expansion for $\tanh^{-1} z$ it is easy to deduce that

$$I_n = 1 - (1/\beta a)(v + v^3/3 + \ldots + v^{n-1}/n - 1) \qquad n \text{ even} \tag{3.10}$$

$$= (1/\beta a)\frac{v^{n+1}}{(n+1)}\left(1 + \frac{n+1}{n+3}v^2 + \frac{n+1}{n+5}v^4 + \ldots\right) \tag{3.11}$$

where $v = \tanh \beta a = \tanh a/kT$ (see figure 2). Each bifurcation occurs at $I_n = \frac{1}{2}$, at increasingly lower temperatures as n increases. The set of bifurcating solutions is shown schematically as a function of temperature in figure 3.

4. Nature of the non-trivial solution close to $K = 0$

In this section we derive an explicit functional form for the stable solution close to and just below the first bifurcation point. The method can be applied to the other

solutions but we have not attempted this in detail. We return to equations (3.4) which can be written as

$$\begin{aligned} K_2M_2 &= -2M_2^2+3M_2M_4-4M_4^2+\ldots \\ K_4M_4 &= 3M_2^2-16M_2M_4+10M_2M_6+3M_4^2+\ldots \\ K_6M_6 &= 15M_2M_4-36M_2M_6-60M_4^2+\ldots \\ K_8M_8 &= 28M_2M_6+35M_4^2+M_8-\ldots. \end{aligned} \tag{4.1}$$

Close to the first bifurcation, K_2 is small and K_n is large for $n>2$. Furthermore, each M_n is close to zero. If we assume $K_2=\varepsilon\approx 0$ and $M_2=\eta\approx 0$ and retain lowest-order terms in η, we find that M_4 is of order η^2, M_6 of order η^3 and so on. To lowest order in η, the equation for M_2 gives

$$(\varepsilon+2\eta)\eta=0$$

which has the solutions $\eta=0$, corresponding to the trivial solution, or $\eta=-\frac{1}{2}\varepsilon$. Since the even moments must be positive, this solution is not physical for $\varepsilon>0$, i.e. above the critical temperature, τ_c, but gives the stable solution below τ_c, i.e. for $\varepsilon<0$. To lowest order in η, the higher moments are given by $M_4=3\eta^2/K_4$, $M_6=15\eta^3/K_6$, $M_8=63\eta^4/K_8,\ldots$, or in general,

$$M_n=(2^{n-2}-1)\eta^{n/2}/K_n \qquad n>2,\ n \text{ even}. \tag{4.2}$$

To obtain a complete functional form for M_n in terms of n, we require an estimate of the K_n. To do this we return to (3.11) from which we can estimate

$$I_n\approx\frac{1}{\beta a(1-v^2)}\frac{v^{n+1}}{(n+1)}.$$

This is in fact an upper bound to I_n. We therefore estimate an asymptotic form for K_n, valid for large n, namely

$$K_n\approx\frac{C(n+1)}{v^{n+1}}$$

where C is a smooth function of temperature at $K_2=0$ and is independent of n. We therefore obtain

$$M_n\approx D(2\sqrt{\eta}v)^n/(n+1) \qquad \text{for } n\geqslant 4$$

where $\eta=-\frac{1}{2}\varepsilon>0$. The moment generating function (3.5) is given by

$$\begin{aligned} \phi(t) &= 1+\eta\frac{(\mathrm{i}t)^2}{2!}+D\sum_{\substack{n\geqslant 4\\ n\,\text{even}}}\frac{(2\sqrt{\eta}v\mathrm{i}t)^n}{(n+1)!} \\ &\approx D\sin(2\sqrt{\eta}vt)/(2\sqrt{\eta}vt). \end{aligned} \tag{4.3}$$

In (4.3) we have approximated terms which are either constant or of order η whose contribution to the probability distribution $\mathcal{M}(x)$ is minor. Inverting (4.3) using (3.6) we obtain

$$\mathcal{M}(x)=\begin{cases} D/(2\sqrt{\eta}v) & \text{for } |x|\leqslant 2\sqrt{\eta}v \\ 0 & \text{otherwise.} \end{cases}$$

The distribution is correctly normalised by putting $D = 1$. Values of $D \neq 1$ arise from the approximations made in the above derivation. Thus the probability of finding a non-zero value of the order parameter is uniform below the critical temperature. Moreover, the possible values increase as $\varepsilon^{1/2}$, below τ_c, which thus defines the analogue of the critical exponent β for this model. We must emphasise however that the analysis given here is only valid for small values of ε.

5. Conclusion

We have studied the random bond Ising model on the $q = 3$ tree. The model has been solved using the method of ring recurrence (McKenzie 1986). We have shown the existence of a trivial solution which corresponds to paramagnetic behaviour above a critical point τ_c, and which becomes unstable below the critical point. There exists an infinite bifurcation set below τ_c, and one obtains a non-trivial solution from each bifurcation. The nature of the non-trivial solution occurring at the highest temperature is investigated in detail and a critical exponent $\beta = \frac{1}{2}$ is deduced. The model is very easily generalised to any symmetric distribution of bond energies and to trees of arbitrary coordination numbers.

References

Abramowitz M and Stegun I A 1965 *Handbook of Mathematical Functions* (New York: Dover)
McKenzie D S 1981 *J. Phys. A: Math. Gen.* **14** 3267
—— 1986 to be published
Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
Thompson C J 1982 *J. Stat. Phys.* **27** 457